Lab 7:
Numerical Solutions for Differential Equations

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Abstract

Techniques like numerical methods for solving differential equations give us the ability to approximate solutions with very high degrees of accuracy. Although such techniques can be used in solving any differential equations, we rely on them most when no analytical solution can be arrived at. It is in such cases that the necessity for approximation techniques like the Euler and Runge-Kutta methods becomes apparent.

1 Introduction

Using nothing more than Newtons law we are able to accurately descrive the motion of various objects. Some of yielded equations of motion can be easily solved analytically. There also exist other, differential equations that are not solvable analytically. For such equations computational techniques must be implemented. The motion of a large amplitude pendulum is one such case, and will be analyzed here. The methods that be implimented are Eulers', Runge-Kutta 2, and Runge-Kutta 4.

In the case of the simple harmonic oscillator we have the force law:

$$\mathrm{F = -dU/dx = -Kx = m\ dv/dt}$$

$$\mathrm{U = 1/2\ K}x^2$$

where F is the force whith spring constant K, displacement x, mass m, velocity v, and simple harmonic oscillator potential U.

In order to use the techniques mentioned above we first rewrite our force law as a set of coupled first order differential equations:

$$dv/dt = -Kx/m \tag{1}$$

$$v = dx/dt \tag{2}$$

Using equations 1 and 2 we are able to implement all three methods of numerical analysis. In the case of a harmonic oscillator we are able to solve its equations analytically:

$$x(t) = x_o cos(wt)$$
$$v(t) = dx/dt = -wx_o sin(wt)$$

Where the angyular frequency w = sqrt(K/m). With solved solutions for both the position and velocity we are able to compare the results obtained using numerical methods to theoretical vaules for each.

The Euler technique uses a recursion algorith. This means, given inital conditions, a physical system can be modeled by predicting the next system state through the use of small intervals. Each subsequent step is based off the last. The downside to such a method is in its accumulation of error. Any derivation from the true values causes a larger error in each following step approximation. This means that once the error begins to grow it does so at a increasing rapid rate.

The Runge-Kutta method improves upon this by accounting for the change in the slope of the funtion is attempts to model. This is to say that the Runge-Kutta method allows us to take acceleration into account. The general form of the Runge-Kutta methods are:

$$y(t + dt) = y(t) + k_2 \tag{3}$$

$$y(t + dt) = y(t) + 1/6 * (k_1 + 2k_2 + 2k_3 + k_4) \tag{4}$$

Here equation 3 is for the Runge-Kutta 2 method and Runge-Kutta 4 uses equation 4. The equations below explicitly state the vaules for k_1-k_4.

$$k_1 = dt * f(t, y(t))$$
$$k_2 = dt * f(t + dt/2, y(t) + k_1/2)$$
$$k_3 = dt * f(t + dt/2, y(t) + k_2/2)$$
$$k_4 = dt * f(t + dt, y(t) + k_3)$$

This general forms are used in modeling our coupled equations.

2 Computational problem

The problem we will be focusing is how to accurately describe harmonic motion. The notions and methods developed here will be expanded upon and generalized. Once generalized these techniques can be used to solve any coupled system. To show this the position of a simple harmonic oscillator will be defined. Once completed we implement these methods to model the motion of a large amplitude pendulum; something not solvable analytically. From Newtow's law we know a simple pendulums' equation of motion to be:

$$mL\frac{d^2\theta}{dt^2} = -mgsin(\theta) \tag{5}$$

where m is the mass of the bob, L is the length, θ is the angle through which is swings, and g is the local gravitational constant (9.8 m/s^2).

In rudimentary pendulum problems the amplitude is chosen to be small enough that the small angle approximation can be exploited:

$$sin(\theta) \approx \theta \tag{6}$$

At angles that are not much small than 1 we are no longer able to use equation 6. This causes our coupled system of equations to be one that is not solvable analytically. It is here that numerical techniques become a necessity.

3 Results

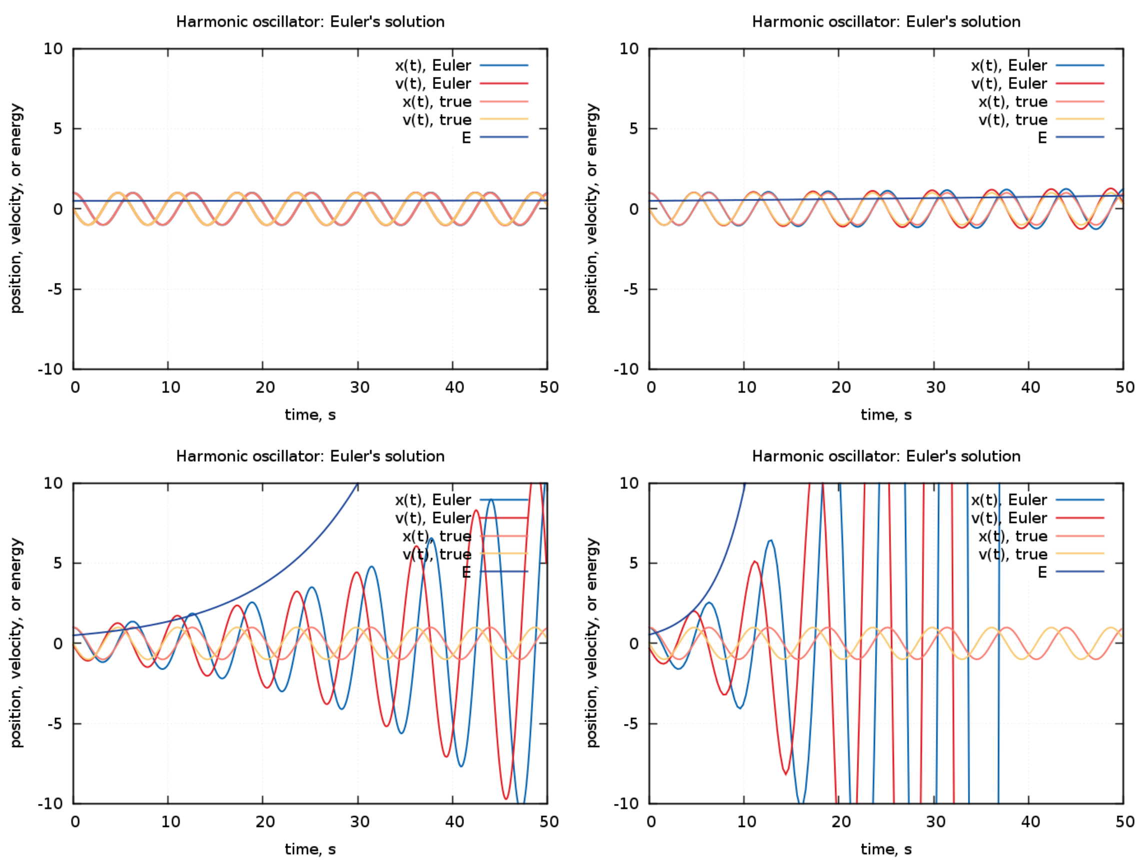

Figure 1: *Position, velocity, and energy of harmonic oscillator, using the Euler method. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

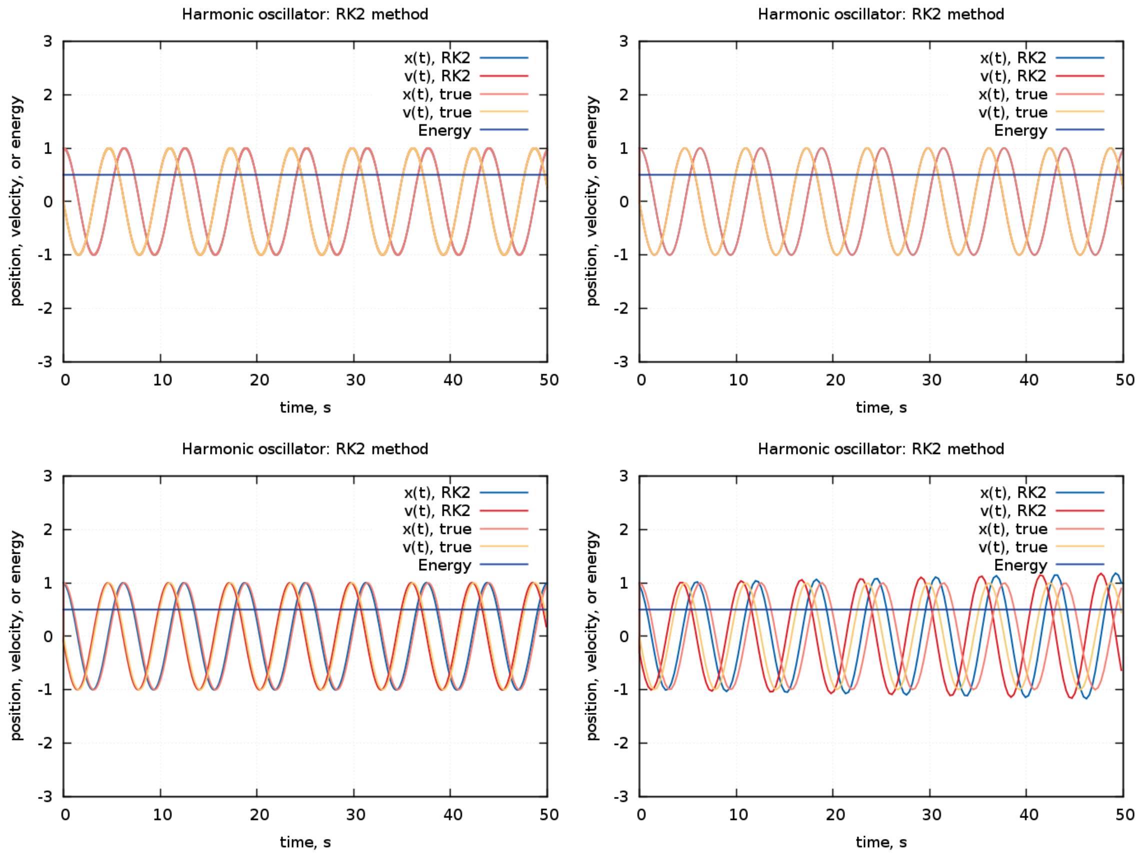

Figure 2: *Position, velocity, and energy of harmonic oscillator, using the Runge-Kutta 2 method. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

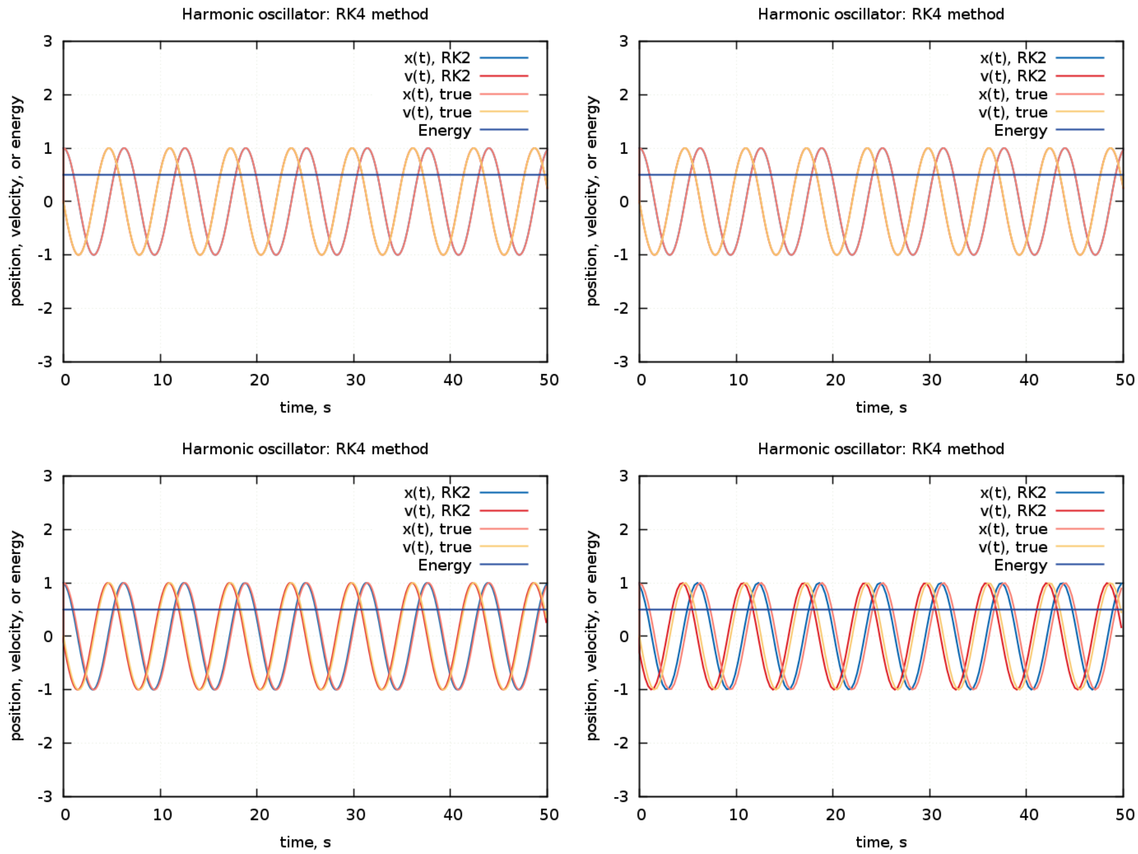

Figure 3: *Position, velocity, and energy of harmonic oscillator, using the Runge-Kutta 4 method. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

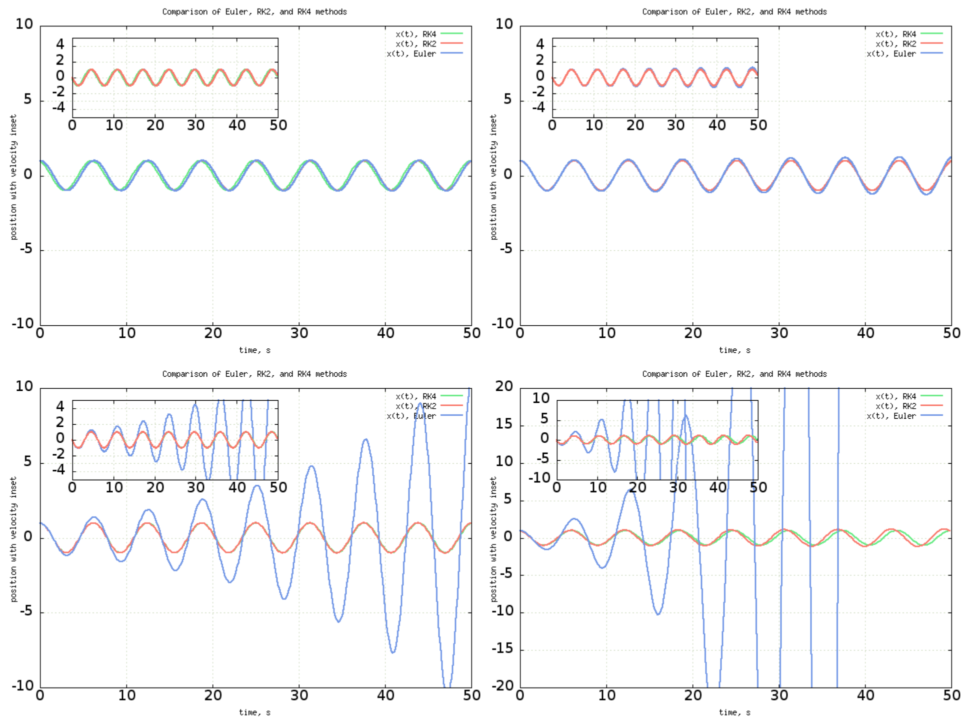

Figure 4: *Comparison of all three methods, for a harmonic oscillator: position shown in the larger graphs, with velocity inset. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

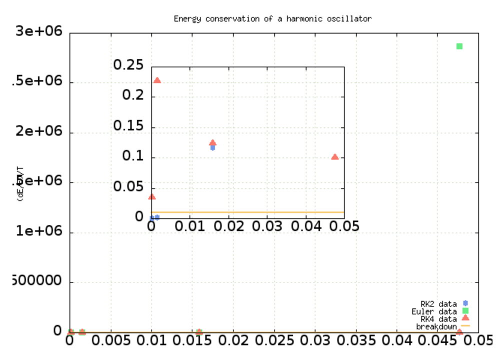

Figure 5: *Fractional energy change in one period of the oscillator vs the fractioon of the oscillator period that the time interval dt fills*

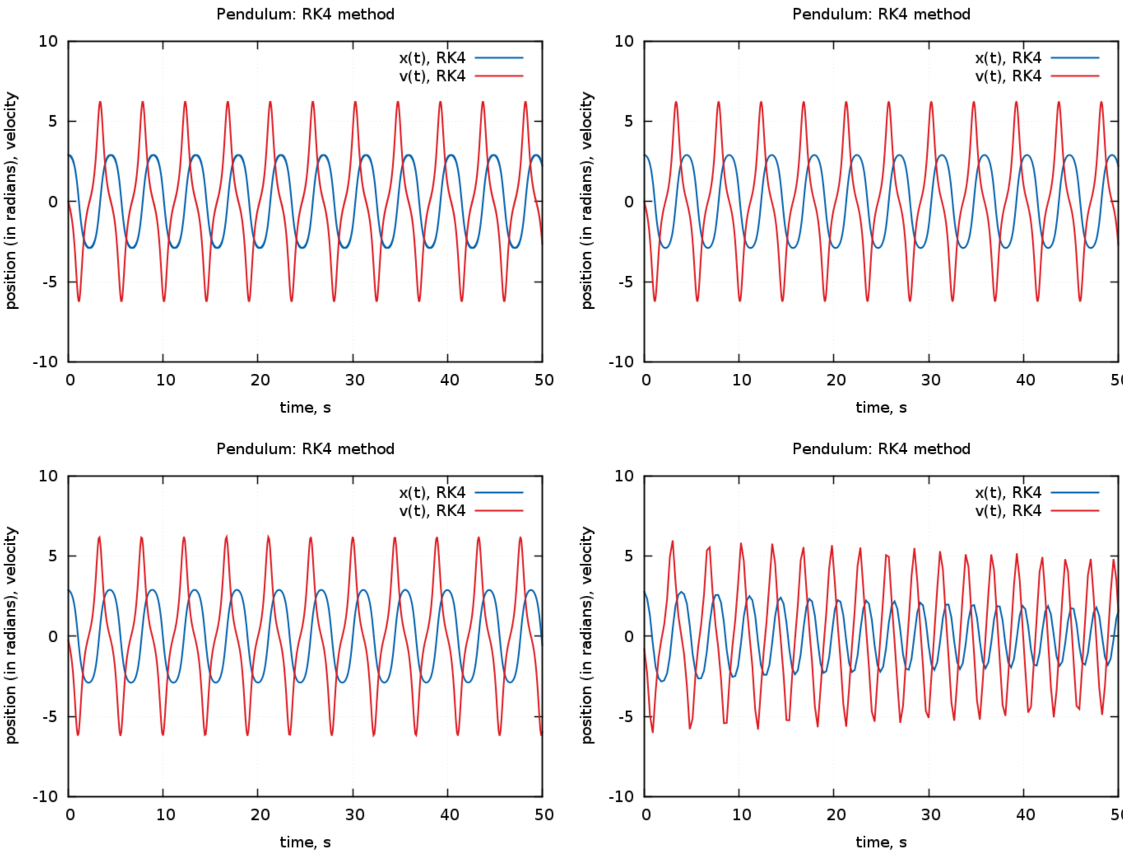

Figure 6: *Position and velocity of a large scale pendulum, using the RK4 method; where x begins at 2.9 and in measured in radians. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

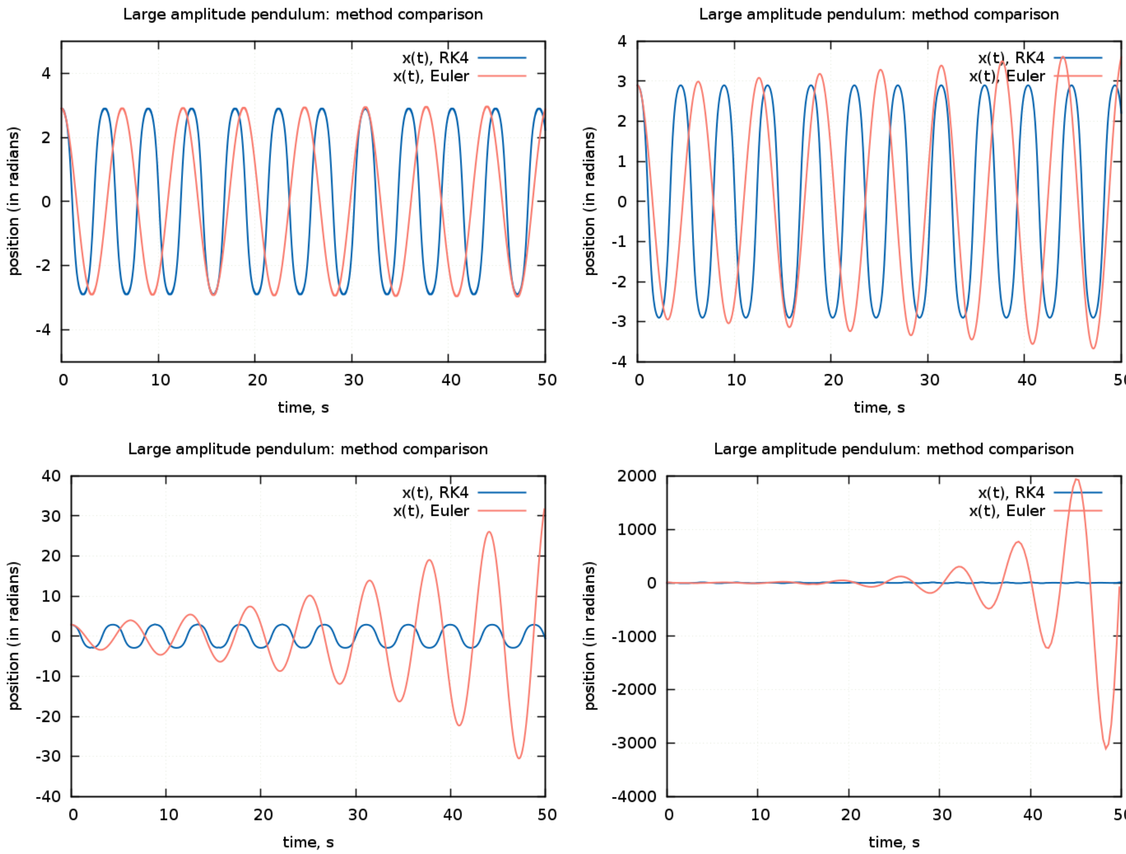

Figure 7: *Position comparison a large scale pendulum using RK4 and Euler methods, where x is measured in radians and begins at 2.9. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

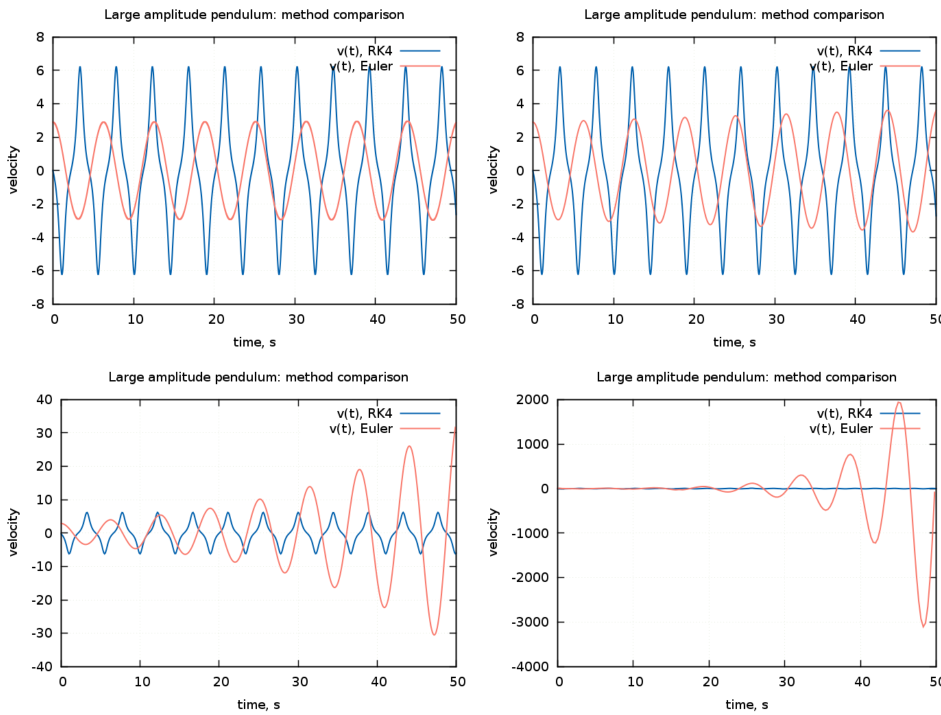

Figure 8: *Velocity vs time comparisison of RK4 Euler methods for a large amplitude pendulum. Top Left: 1ms, Top Right: 10ms, Bottom Left: 100ms, Bottom Right: 300ms*

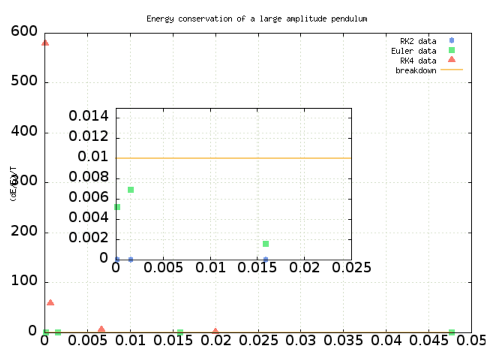

Figure 9: *Fractional energy change in one period of the pendulum vs the fractioon of the pendulum period that the time interval dt fills*

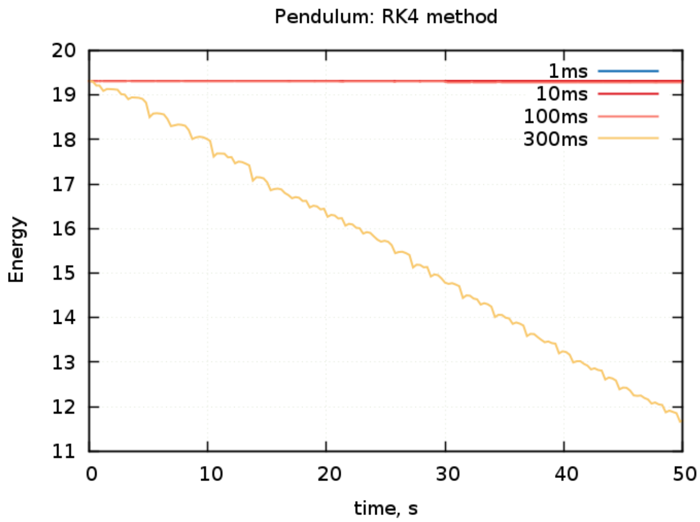

Figure 10: *Energy of a large amplitude pendulum, using the Runge-Kutta 4 method*

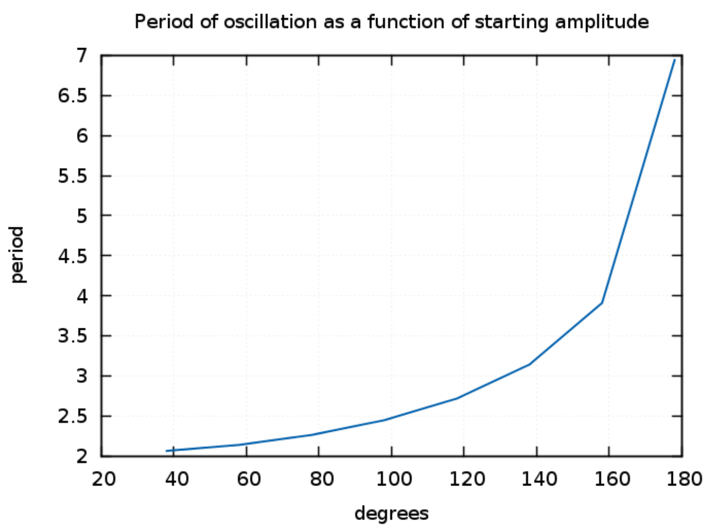

Figure 11: *Period of the oscillation as a function of the starting amplitude; in 20 degree increments*

4 Analysis

Figure 1 shows the position, velocity, and energy for the harmonic oscillator, using the Euler method. As shown in the graphs, the system seems to always be gaining energy. This apparent gain of energy is due to the approximation used in determining the position and velocity at each point. As the time interval increases, from 1-300 ms, the energy increases at a faster rate. When larger interval steps the collected error grows faster. This is also diplayed by the position and velocity curves themselves. As the time interval increases the disparity between calculated and theoretical values of position and velocity increases. Figure 2 graphically represents how well the Runge-Kutta 2 method approximates the position and velocity of a harmonic oscillator. As shown by the graph, very little energy is gained by this system. Even as the time interval is increased the Runge-Kutta 2 method holds tightly to the true values for position and velocity. The same motion is described in figure 3, but using the Runge-Kutta 4 method. The Runge-Kutta 4 method is the most accurate of the three techniques implimented for analysing the motion of a harmonic oscillator. As shown in figure 3, the Runge-Kutta 4 method is identical to the theoretical values, for time intervals of 10 ms or less. This method is better than the other two used, with regards to conservation of energy.

Figure 4 shows a side by side comparison of all three methods. Starting at 1 ms the Euler method yields different results, than either of the other two methods used. Here the Runge-Kutta 2 and Runge-Kutta 4 methods are shown to be nearly identical until the time interval is increased to 300 ms. This tells us that for properly sized time steps either method will generate accurate results. However, it is clear from these graphs that the Runge-Kutta 4 method is superior to both the Runge-Kutta 2 and Euler method. Figure 5 shows that the frational change in energy per period of the harmonic oscillator is held at its lowest when using the Runge-Kutta 2 method. Figure 5 shows that the maximum interval dt can be, in order to conserve energy, is 10 ms. This is using the data points generated. Had it been possible to fit a curve to the data set a more accurate maximum interval could have been calculated. The graph indicates it to be less than 100 ms but still slightly more than 10ms.

Figure 6 shows an analysis of the position and velocity of a pendulum, using the Runge-Kutta 4 method. As shown in the graph using a time step of 300 ms, the system loses energy. This apparent loss of energy is due to the error introduced by this numerical technique. In order to have accurate results, small time intervals must be used. This is the case, even though the Runge-Kutta 4 method is better at modeling the large amplitude pendulums motion that the Euler method (as shown by figures 7 and 8). Figure 9 shows that the Runge-Kutta 4 method is least favorable, when viewed from an energy conservation persepective. This graph shows that using the Runge-Kutta 4 method means we assume the system to have an overall loss of energy. Figure 10 shows where the Runge-Kutta 4 method fails. For time intervals less than that of 300 ms the energy of the system holds constant. Only at the barrier does the Runge-Kutta 4 method fail, by showing an overall lose of energy. Even though the Runge-Kutta 4 method breaks down for large time intervals it is still shown to be the best, of the three numerical methods analyzed here, at modeling the position and velocity of a large amplitude pendulum. The period of oscilation is not constant due to the sine term in our mathematical model of this pendulums motion. This can be seen in figure 11. Here the period is shown to be increasing.

5 Conclusion

Systems of differential equations are a necessary tool in modeling the motion of any object with a force acting on it. In many cases these equations are not solvable analytically. It is here that we need tools such as numerical solutions. Methods, such as Runge-Kutta and Euler, exist and aid us in modeling such systems. Without such techniques the amount of time it would take to arrive at an accurate representation of an objects motion would grow factorial; granted it could ever be achieved analytically.

6 Extra Credit

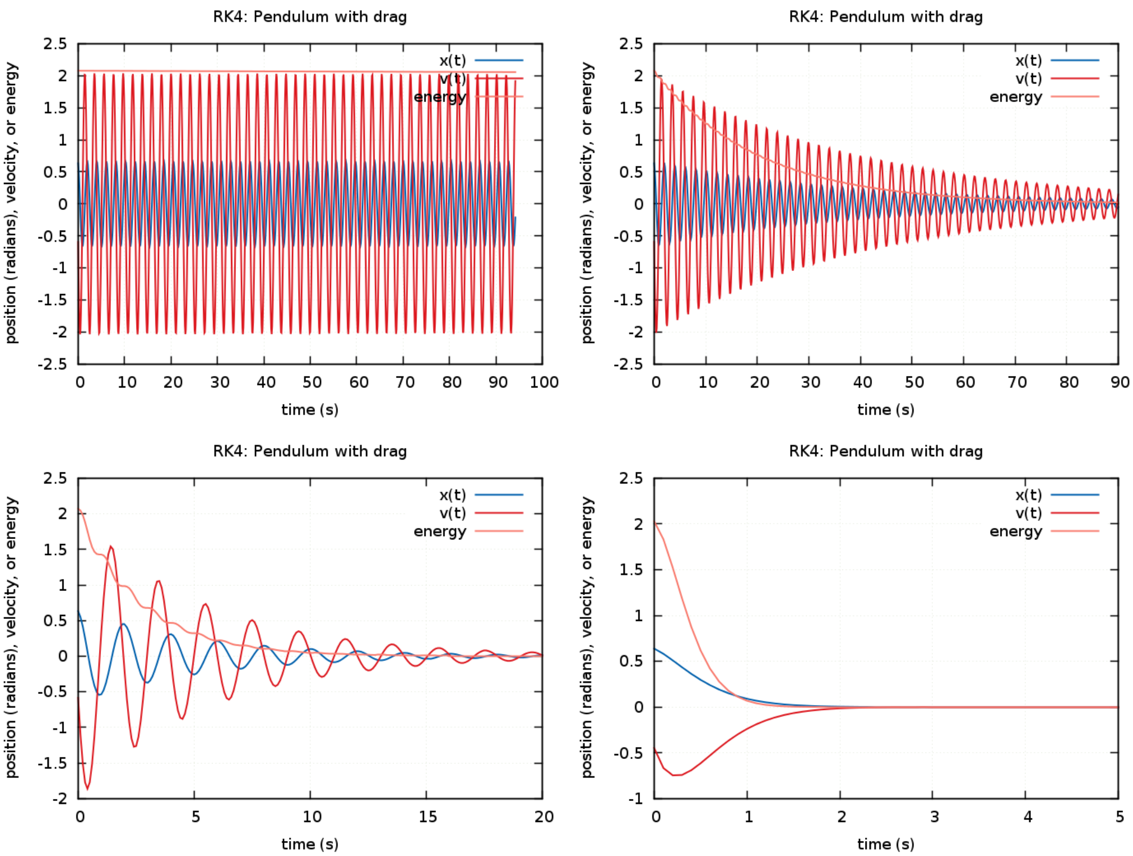

Figure 12: *Velocity vs time co Top Left: b=0 kg/s, Top Right: b=0.05 kg/s, Bottom Left: b=1/e kg/s, Bottom Right: b=6 kg/s*

The top left graph in figure 11 shows energy conservation for the case of a large scale pendulum, using the Runge-Kutta 4 method. This was done by adding a drag force term to the program and setting the drag coefficient, b, to zero. Once this was shown to work, various other values of b can be utilized. The under-damped case occurs at b=0.05 kg/s and is shown in the top right of figure 11. The case of over-damping occurs when b=1/e k kg/s, and is shown by the bottom left graph in figure 11. Finally, when b=6 kg/s the critically damped case occurs. This is shown by the bottom right graph in figure 11.

References

[1] R. H. Landau and M. J. Paez, "Computational Physics, Problem Solving with Computers," (Wiley: New York) 1997.

[2] Gorham, Peter. "Physics 305 Differential Equations." P305lab7. Phys.hawaii.edu, 6 March 2015. Web. 8 Mar. 2015.

[3] Allain, Alex. "Programs as Data: Function Pointers." Function Pointers in C and C++. Cprogramming, n.d. Web. 6 Mar. 2015.

Appendix

Various programs and plot files used in generating the graphs and data in this paper can be found online at:
`http://www2.hawaii.edu/~cmutnik/lab7.html`